Subjective expected utility

Theory of Individual and Strategic Decisions

Lecture 2b
Neuroeconomics: An interdisciplinary approach to how the brain makes us deciding

Fall 2019

Roadmap

Objective vs. subjective uncertainty

- Up to now, uncertainty was modeled by lotteries (risk or objective uncertainty).
- What if uncertainty is more complex (as often the case in real life)?
    - Examples: Random experiments like a horse race, the next elections, the price of a stock, a car accident, the weather etc.
    - Here the probabilities are not "objectively known" to the decision maker. The decision maker has to form **subjective beliefs**.

Modeling subjective uncertainty

- We start with the lotteries $\Delta(Z)$ over Z.
- **States.** The set of states is a finite set Ω of realizations of some random experiment with unknown probabilities.
  - Example: if our random experiment is "a car accident", then the two states are:

$$\omega_1 : \text{to have a car accident,}$$
$$\omega_2 : \text{not to have a car accident.}$$

- **Acts.** An act is a function $f : \Omega \to \Delta(Z)$.
  - An act is a contingent plan of lotteries, one for each state.
  - Example: if the random experiment is a car accident, then an act can be thought as a **contract** that yields one lottery at ω_1 (if there is an accident) and another lottery at ω_2 (if there is no accident).

An insurance example

Example

An agent wants to insure his car which is worth 1000 Euros. The insurance company offers the following contract at the price of 100 Euros: They cover the whole damage if there is an accident (at state ω_1), whereas they do not pay anything if there is no accident (at state ω_2). This means that the agent has the choice between two acts, "to buy the insurance" (act f) or "not to buy the insurance" (act g):

act	interpretation	accident (ω_1)	no accident (ω_2)
f	insurance	900	900
g	no insurance	0	1000

It is crucial to notice here that the agent does *not know the probability* of an accident occurring.

An example of free-riding

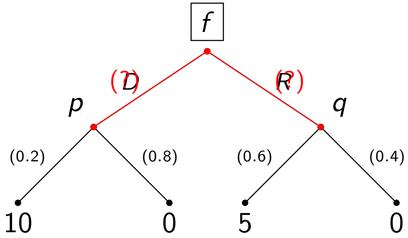
Example

A passenger is thinking whether he should buy a train ticket for the trip he is about to take. The ticket costs 20 Euros and if he gets caught without a ticket he will pay a fine of 200 Euros. The railway's policy against free-riding is based on *random checks*: the inspector checks the ticket of every other passenger, i.e., if the inspector enters the train the probability of getting caught is 0.5. This is known to the passenger. However, the passenger does not know the probability of an inspector entering the train. There are two states, one where there is inspection (ω_1), and one where there is no inspection (ω_2). Thus, the passenger has the choice between "buying a ticket" (act f) and "not buying a ticket" (act g):

act	interpretation	inspection (ω_1)	no inspection (ω_2)
f	ticket	-20	-20
g	no ticket	$(0.5 \circledast 0, 0.5 \circledast -200)$	0

Acts as compound lotteries

- An act can be thought as a compound lottery, where the probabilities of the first round are not given.
- Let $\Omega = \{\text{Dem } (D), \text{Rep } (R)\}$, and f is a contract (act) such that
    - $f(D) = p \in \Delta(Z)$
    - $f(R) = q \in \Delta(Z)$



- An act cannot be written as a standard lottery, because the probabilities of D and R are not given.

The set of acts

- The **set of acts** is the set of all functions $f : \Omega \to \Delta(Z)$.
- We denote the set of acts by $\mathcal{F}$.
- An act can be thought as a vector of lotteries:

$$\mathcal{F} := \underbrace{\Delta(Z) \times \cdots \times \Delta(Z)}_{\text{one for each state in } \Omega}$$

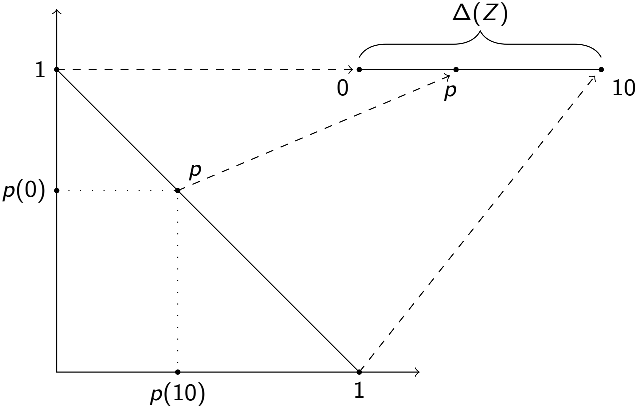
Thus, an act $f \in \mathcal{F}$ is often written as

$$(f(\omega_1), \ldots, f(\omega_m)).$$

- A **constant act** $(p, \ldots, p)$ induces the same lottery p at every state, e.g., "buying the insurance" or "buying a ticket".
- The set of constant acts is denoted by $\mathcal{F}_c$.

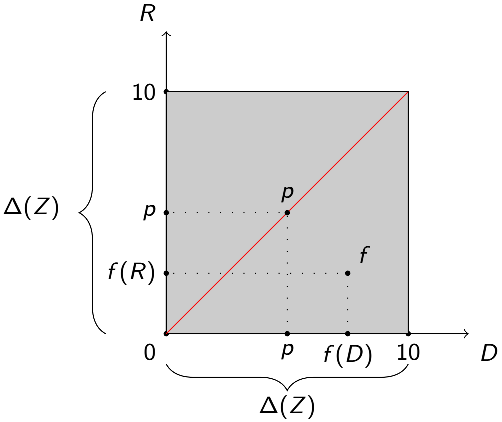
Graphical representation of the set of acts

- Let $Z = \{0, 10\}$ and $\Omega = \{D, R\}$.
- Recall that $\Delta(Z)$ can be represented by a linear segment:



Graphical representation of the set of acts

- Since we have two states, we need two such segments (of equal length), one for state D and one for state R.
- Thus, the space of acts is the shaded square.
- The space of constant acts is the diagonal.



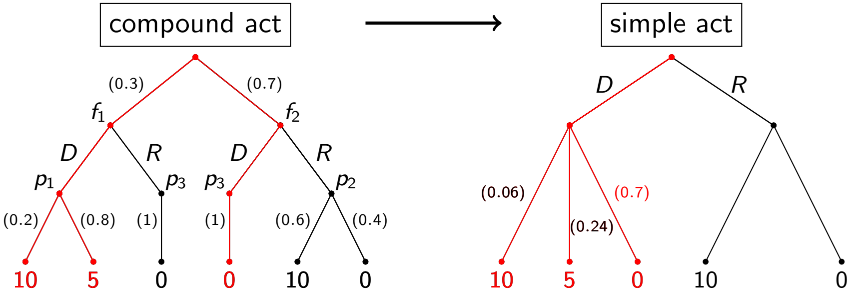
Compound acts

- A **compound act** is a lottery over acts.
- The compound act $(\alpha_1 \circledast f_1, \dots, \alpha_\ell \circledast f_\ell) \in \Delta(\mathcal{F})$ is reduced to the simple act

$$(\alpha_1 \circledast f_1, \dots, \alpha_\ell \circledast f_\ell)(\omega)(z) = \alpha_1 \cdot f_1(\omega)(z) + \cdots + \alpha_\ell \cdot f_\ell(\omega)(z)$$

Graphical illustration of compound acts

- Consider the lotteries $p_1 = (0.2 \circledast 10, 0.8 \circledast 5)$, $p_2 = (0.6 \circledast 10, 0.4 \circledast 0)$ and $p_3 = (1 \circledast 0)$.
- Let $\Omega = \{D, R\}$, and $f_1 = (p_1, p_3)$ and $f_2 = (p_3, p_2)$.



- $(0.3 \circledast f_1, 0.7 \circledast f_2)(D)(10) = 0.3 \cdot 0.2 = 0.06$
- $(0.3 \circledast f_1, 0.7 \circledast f_2)(D)(5) = 0.3 \cdot 0.8 = 0.24$
- $(0.3 \circledast f_1, 0.7 \circledast f_2)(D)(0) = 0.7 \cdot 1 = 0.7$

Graphical illustration of compound acts

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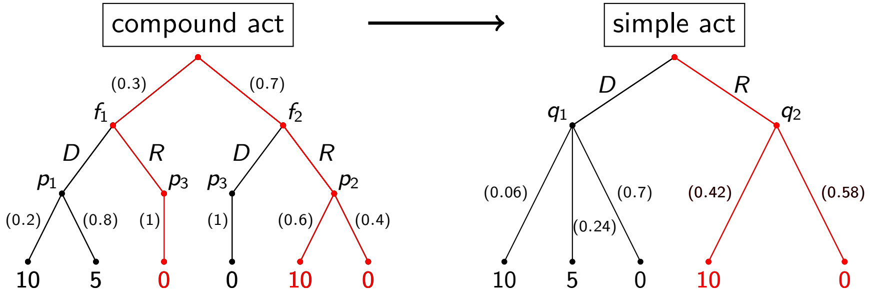


- $(0.3 \circledast f_1, 0.7 \circledast f_2)(R)(10) = 0.7 \cdot 0.6 = 0.42$
- $(0.3 \circledast f_1, 0.7 \circledast f_2)(R)(0) = 0.3 \cdot 1 + 0.7 \cdot 0.4 = 0.58$
- $(0.3 \circledast f_1, 0.7 \circledast f_2) = (q_1, q_2)$ ... simple act

Roadmap

1. States and Acts
2. Preferences over acts
3. Subjective Expected Utility

Acts as alternatives

- We take the set of acts to be the set of alternatives, i.e.,

$$\boxed{X := \mathcal{F}}$$

- Then, we consider a decision maker with preferences over $\mathcal{F}$ still denoted by $\succsim$.
- Whenever we write $p \succsim q$, we mean

$$(p, \ldots, p) \succsim (q, \ldots, q).$$

Anscombe-Aumann axioms

Definition

We say that an agent has **Anscombe-Aumann (AA) preferences** over $\mathcal{F}$ whenever $\succsim$ satisfies the following axioms:

(A_1'') **Completeness**: For any two acts $f, g \in \mathcal{F}$, it is the case that $f \succsim g$ or $g \succsim f$.

(A_2'') **Transitivity**: For any three acts $f, g, h \in \mathcal{F}$ with $f \succsim g$ and $g \succsim h$, it is the case that $f \succsim h$.

(A_3'') **Continuity**: For any three acts $f, g, h \in \mathcal{F}$ with $f \succ g \succ h$, there exists some $\alpha \in (0,1)$ such that $g \sim \big(\alpha \circledast f, (1-\alpha) \circledast h\big)$.

(A_4'') **Independence of irrelevant alternatives (IIA)**: For any three acts $f, g, h \in \mathcal{F}$ and any $\alpha \in [0,1]$, it is the case that $f \succsim g$ if and only if $\big(\alpha \circledast f, (1-\alpha) \circledast h\big) \succsim \big(\alpha \circledast g, (1-\alpha) \circledast h\big)$.

(A_5'') **Monotonicity**: For any two acts $f, g \in \mathcal{F}$ with $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$ it is the case that $f \succsim g$.

(A_6'') **Non-triviality**: There exist two acts $f, g \in \mathcal{F}$ such that $f \succ g$.

We have two additional axioms

- **Monotonicity** says that if *at every single state* the agent weakly prefers the lottery induced by f over the lottery induced by g, then she weakly prefers the act f over the act g.
- In other words, if the agent weakly prefers f over g *regardless of which state is realized*, then the agent weakly prefers f over g all together without worrying about which state is realized (uncertainty over the states becomes irrelevant).
- **Non-triviality** says that there exist at least two acts between which the agent is *not* indifferent.
- Without this axiom, the agent would not care about anything, and the problem becomes uninteresting to study.

Roadmap

Representing AA preferences

Can we represent AA preferences with a utility function?

- Again, the answer does not follow directly from existence of a utility function over (certain) outcomes. Recall,

Proposition

If preferences are a weak order over a finite set of alternatives, then there exists a utility representation.

- We cannot use this result because the set of acts $\mathcal{F}$ is not finite.

Subjective expected utility theorem

Theorem (Anscombe & Aumann, 1963)

Consider a finite set of outcomes Z and a finite state space Ω. Then, the preferences $\succsim$ over $\mathcal{F}$ satisfy $A_1'' - A_6''$ if and only if there is a function $u : Z \to \mathbb{R}$ and a probability measure $\mu \in \Delta(\Omega)$ such that for every $f, g \in \mathcal{F}$,

$$f \succsim g \;\Leftrightarrow\; \sum_{\omega \in \Omega} \mu(\omega) \left(\sum_{z \in Z} f(\omega)(z) u(z) \right) \geq \sum_{\omega \in \Omega} \mu(\omega) \left(\sum_{z \in Z} g(\omega)(z) u(z) \right)$$

Moreover, μ is unique and $u : Z \to \mathbb{R}$ is unique up to a positive linear transformation.

Subjective expected utility function

The **subjective expected utility** function that represent $\succsim$ is

$$U(f) := \sum_{\omega \in \Omega} \mu(\omega) \cdot \left(\sum_{z \in Z} f(\omega)(z) \cdot u(z) \right)$$

$\mu(\omega)$ represents **subjective beliefs** about the likelihood of the states.
$(f(\omega_1), \ldots, f(\omega_m))$ is **act** written as **vector of lotteries**, one for each state.

The subjective expected utility function

- The construction of the subjective expected utility (SEU) function is done in two steps.
  - First, we get a **utility function**, $u : Z \to \mathbb{R}$.
  - Second, we get **subjective beliefs**, $\mu \in \Delta(\Omega)$.
- These two steps correspond to the two expectations that we need to take in order to calculate the SEU of an act $f \in \mathcal{F}$:

$$U(f) := \sum_{\omega \in \Omega} \mu(\omega) \cdot \underbrace{\left(\sum_{z \in Z} f(\omega)(z) \cdot u(z) \right)}_{\text{weighted sum over outcomes at } \omega}$$

- For each ω, we compute the vNM expected utility $U(f(\omega))$ of the lottery $f(\omega) \in \Delta(Z)$.
- Then, we take a second (subjective) expectation over the states and $U(f)$ becomes:

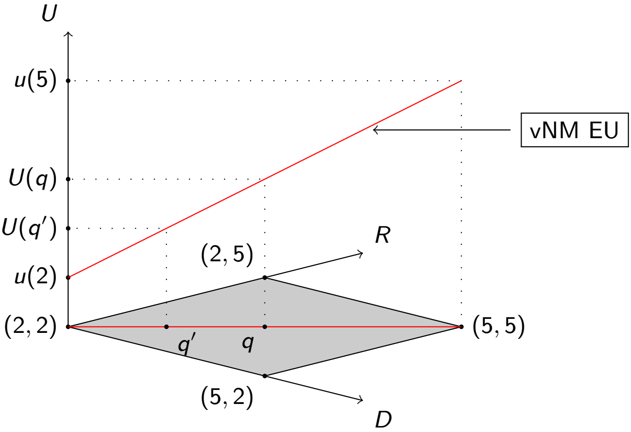
$$U(f) := \underbrace{\sum_{\omega \in \Omega} \mu(\omega) \cdot U\big(f(\omega)\big)}_{\text{weighted sum over states}}$$

The subjective expected utility function

- The utility function over constant acts is obtained in the same way as in vNM's expected utility theorem.
- This is because the AA axioms over the constant acts can be seen as vNM axioms over the lotteries.

The subjective expected utility function

- Let $Z = \{2, 5\}$ and $\Omega = \{D, R\}$.
- Draw the set of acts and the set of constant acts.
- Assume, $5 \succ 2$. Hence, $u(5) > u(2)$.
- Subjective probabilities are not involved in this part.

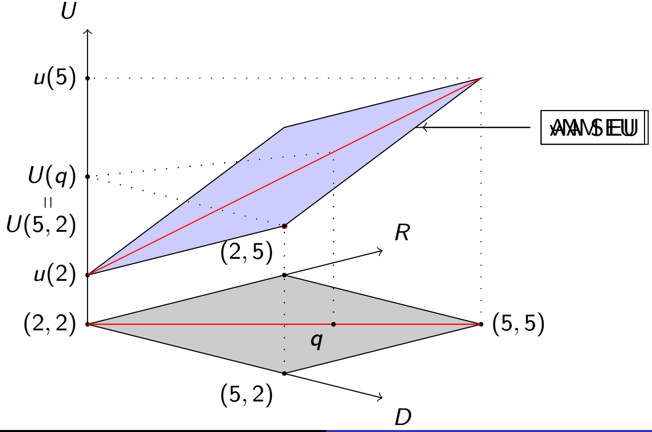


The subjective beliefs

- The subjective beliefs are given by the preferences of the agent over non-constant acts.
- The idea is as follows:
  - Take a non-constant act that induces a “good lottery” at one state and a “bad lottery” at another state.
  - By continuity, the agent can be made indifferent between getting this act and getting a constant act that gives *at every state* the “good lottery” with some probability and the “bad lottery” with the remaining probability.
  - This probability can then be interpreted as the subjective belief over the state where the “good lottery” is obtained.

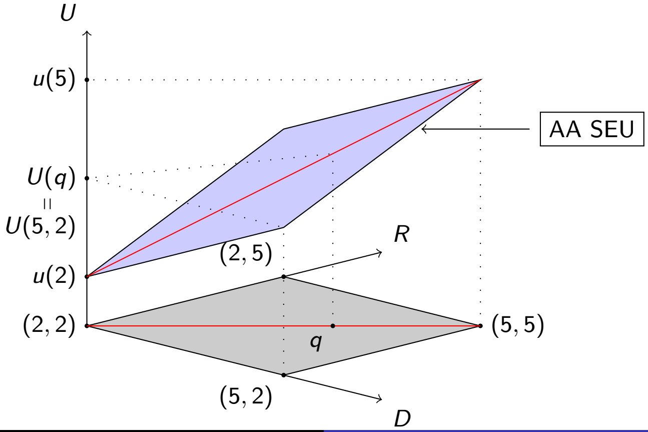
The subjective beliefs

- Assume, $(5,5) \succ (5,2) \succ (2,2)$.
- By A_3'' (continuity), there is some $q \in \mathcal{F}_c$ s.t. $(5,2) \sim q$.
- Hence, $U(5,2) = U(q)$.
- This gives a new point, and the utility representation.



The subjective beliefs

- There is $\alpha \in (0,1)$ s.t. $q = \big(\alpha \circledast (5,5), (1-\alpha) \circledast (2,2)\big)$.
- q gives 5 with prob α and 2 with prob $1-\alpha$.
- $(5,2)$ gives 5 with prob $\mu(D)$ and 2 with prob $1-\mu(D)$.
- Hence, $q \sim (5,2) \Leftrightarrow \mu(D) = \alpha$.



Importance and limitations of SEU

- The **Subjective Expected Utility Theorem** is a very important advancement in decision theory.
- This is the first result in the literature to directly link **preferences** and **subjective beliefs**.
- There are two (complementary) ways to read this result:
  - The decision maker has preferences over acts, and her/his beliefs follow from these preferences.
  - The decision maker has beliefs over states, and her/his preferences over acts are consistent with these beliefs.
- Theoretically, it does not matter whether preferences or beliefs "come first". It is important that one determines the other *uniquely*.

  From a psychological and neuroeconomis perspective it is, however, interesting to know what comes first.
- Similarly to vNM Expected Utility, SEU does not always accurately describe human behavior. Axioms are sometimes violated.

Violation of IIA: The Ellsberg paradox

- Consider an urn containing three balls. Exactly one ball is **red** and the remaining two balls are either (i) both **black**, or (ii) one is **black** and the other one is yellow, or (iii) both yellow.
- First, note that there are three states:
  - ω_1 : the urn contains 1 red, 2 black and 0 yellow balls
  - ω_2 : the urn contains 1 red, 1 black and 1 yellow balls
  - ω_3 : the urn contains 1 red, 0 black and 2 yellow balls
- Choose between two acts:
  - f_1 : if the winning ball is **red** you get 10 Euros
  - f_2 : if the winning ball is **black** you get 10 Euros
- Choose between two acts:
  - g_1 : if the winning ball is **red** or yellow you get 10 Euros
  - g_2 : if the winning ball is **black** or yellow you get 10 Euros

Violation of IIA: The Ellsberg paradox

- f_1 : if the ball is **red** you get 10 Euros
- f_2 : if the ball is **black** you get 10 Euros
- g_1 : if the ball is **red or yellow** you get 10 Euros
- g_2 : if the ball is **black or yellow** you get 10 Euros

act	2 black/0 yellow (ω_1)	1 black/1 yellow (ω_2)	0 black/2 yellow (ω_3)
f_1	$(1/3 \circledast 10, 2/3 \circledast 0)$	$(1/3 \circledast 10, 2/3 \circledast 0)$	$(1/3 \circledast 10, 2/3 \circledast 0)$
f_2	$(2/3 \circledast 10, 1/3 \circledast 0)$	$(1/3 \circledast 10, 2/3 \circledast 0)$	$(1 \circledast 0)$
g_1	$(1/3 \circledast 10, 2/3 \circledast 0)$	$(2/3 \circledast 10, 1/3 \circledast 0)$	$(1 \circledast 10)$
g_2	$(2/3 \circledast 10, 1/3 \circledast 0)$	$(2/3 \circledast 10, 1/3 \circledast 0)$	$(2/3 \circledast 10, 1/3 \circledast 0)$

- If IIA holds, then it must be $f_1 \succ f_2 \Leftrightarrow g_1 \succ g_2$. In experiments we typically see $f_1 \succ f_2$ and $g_2 \succ g_1$.
- Hence, IIA is **systematically** violated.

Violation of IIA: The Ellsberg paradox

- f_1 : if the ball is **red** you get 10 Euros
- f_2 : if the ball is **black** you get 10 Euros
- g_1 : if the ball is **red or yellow** you get 10 Euros
- g_2 : if the ball is **black or yellow** you get 10 Euros

- Let $u(color\ win) = 1$ and $u(color\ lose) = 0$; q $(2/3 - q)$ belief that balls are black (yellow)
- $f_1 \succ f_2$: $1/3 \times u(R\ win) + q \times u(B\ lose) + (2/3 - q) \times u(Y\ lose) > 1/3 \times u(Red\ lose) + q \times u(B\ win) + (2/3 - q) \times u(Y\ lose)$, that is $1/3 \times 1 > q \times 1$ or $1/3 > q$.
- $g_1 \prec g_2$: $1/3 \times u(R\ win) + q \times u(B\ lose) + (2/3 - q) \times u(Y\ win) < 1/3 \times u(Red\ lose) + q \times u(B\ win) + (2/3 - q) \times u(Y\ win)$, that is $1/3 \times 1 + (2/3 - q) \times 1 < q \times 1 + (2/3 - q) \times 1$ or $1/3 < q$.
- Thus, $f_1 \succ f_2$ and $g_2 \succ g_1$ decision maker has inconsistent beliefs: $q < 1/3$ and $q > 1/3$.

The end.