Preferences and Utility

Economic Theory of Individual Decisions

Lecture 1
Neuroeconomics: An interdisciplinary approach to how the brain makes us deciding

Fall 2019

Roadmap

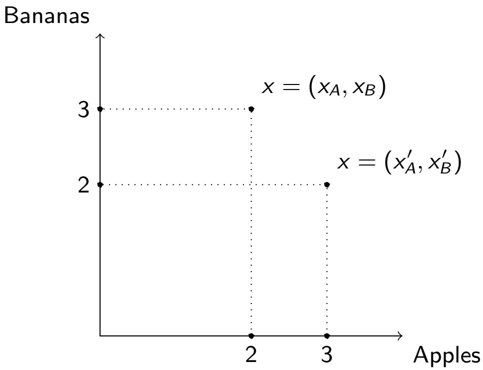
The elements of Decision Theory

The first building block is **preferences** over **alternatives**.

- **Alternatives:** A set X containing everything that the agent could end up with.
  - Examples: a consumption bundle, a lottery ticket, a contingent pension plan, etc.
- **Preferences:** For each pair of alternatives, which of the two alternatives the agent would rather like to have.

An example of the alternatives in consumer theory

- There are two goods, *Apples* and *Bananas* (measured in kg).
- An alternative $x \in X$ is a pair of quantities (x_A, x_B), e.g., a bundle consisting of 2 kg of Apples and 3 kg of Bananas or 3 kg of Apples and 2 kg of Bananas.



Preference relations

- Preferences are formally modeled with a binary **preference relation**, $\succsim$.
- For any pair (x, y) of alternatives, $x \succsim y$ means

x is **weakly preferred** over y.

- A weak preference relation can be thought of as the list of answers to all the questions of the form:

Do you find x to be at least as good as y?

Preference relation as a list of answers

Example

Let the set of alternatives be
$X = \{$"10 Euros", "a national lottery ticket", "a vacation week"$\}$.
For each pair of alternatives we ask the agent *"do you find x at least as good as y?"* The answers are the ones depicted below:

$\succsim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	✓	✓	✓

That is, for instance Money $\succsim$ Money while Money $\not\succsim$ Vacation.

- **Remark:** The following two questions are different:
  - do you find x at least as good as y?
  - do you find y at least as good as x?

  This is because the answer to the first question does not necessarily give you the answer to the second.

From ‘weak’ to ‘strong’ preferences to ‘indifference’

- Starting from the weak preference relation $\succsim$, we can say the following:
  - $x \not\succsim y$ means “x is **not preferred** to y”.
  - $x \succ y$ means that “x is **strictly preferred** to y”. Formally, $x \succ y \Leftrightarrow y \not\succsim x$.
  - $x \sim y$ means that “the agent is **indifferent** between x and y”. Formally, $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$.
- That is, once we know $\succsim$, we can unambiguously derive both $\succ$ and $\sim$.

From weak to strict preference

- Take the weak preference relation from the previous example.
- Derive the strict preference relation, recalling that

$$x \succ y \Leftrightarrow y \not\succsim x$$

- **Remark:** We can go back and forth.

$\succsim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	✓	✓	✓

$\succ$	Money	Lottery	Vacation
Money	X	X	X
Lottery	X	X	X
Vacation	✓	✓	X

From weak preference to indifference

- Take the weak preference relation from the previous example.
- Derive the indifference relation, recalling that

$$x \sim y \Leftrightarrow x \succsim y \text{ and } y \succsim x$$

- **Remark:** We cannot go back and forth.

$\succsim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	✓	✓	✓

$\sim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	X	X	✓

Unreasonable preferences

- So far, we have not imposed any restriction on $\succsim$.
- That is, in principle, we could have a preference relation such that the following two hold simultaneously:
  - The agent *does not find x at least as good as y*.
  - The agent *does not find y at least as good as x*.
- In other words, the agent's preferences are such that
  - $x \not\succsim y$, and hence $y \succ x$,
  - $y \not\succsim x$, and hence $x \succ y$.
- Intuitively, these preferences are not reasonable.
- Hence, we impose some mild restrictions, in order to rule out preferences that do not make sense intuitively.
- These restrictions are called **axioms**.

Axioms of rationality (weak order)

Definition

We say that the preferences of an agent are **rational** (a **weak order**) if the following two axioms are satisfied:

(A_1) **Completeness:** For any two alternatives $x, y \in X$, it is the case that $x \succsim y$ or $y \succsim x$.

(A_2) **Transitivity:** For any three alternatives $x, y, z \in X$ with $x \succsim y$ and $y \succsim z$, it is the case that $x \succsim z$.

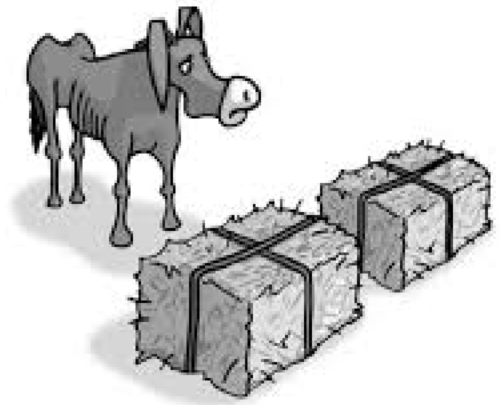
Rational preferences

- Take the weak preference relation from the previous example.
- This preference relation is rational (a weak order).
  - It satisfies completeness: $M \succsim L$, $V \succsim M$ and $V \succsim L$.
  - It satisfies transitivity: $V \succsim M$ and $M \succsim L$ imply $V \succsim L$.

$\succsim$	Money	Lottery	Vacation
Money	✓	✓ (circled)	X
Lottery	✓	✓	X
Vacation	✓ (circled)	✓ (circled)	✓

Violation of completeness

Example (Buridan's ass)



https://www.youtube.com/watch?v=b0p1_LIYvmk

Violation of transitivity I

If alternatives have several characteristics, it may be easy to rank them rationally on each characteristic but not to evaluate the overall desirability of one alternative over the other.

Example (Condorcet preferences)

Let an alternative correspond to a bundle of three goods, Apples, Bananas and Carrots. Now, consider three bundles $x, y, z \in X$ with the corresponding quantities of each fruit:

	Apples	Bananas	Carrots
x	10	20	30
y	30	10	20
z	20	30	10

Now suppose that the decision maker strictly prefers an alternative over another if the corresponding bundle contains strictly larger quantity of at least two fruits. Then, observe that these preferences are such that $x \succ y \succ z \succ x$, implying that they violate transitivity.

Violation of transitivity II

If a decision maker is insensitive to small differences this may lead to intransitivity in the large.

Example (Grains of sugar)

Let the set of alternatives be $X = \mathbb{N}$. Each alternative corresponds to the number of grains of sugar in a cup of coffee. Assume that a spoon of sugar contains 200 grains, and consider an agent who is indifferent between two cups of coffee that differ only by one grain of sugar, but still strictly prefers her coffee with one spoon of sugar compared to no sugar at all. Observe that these preferences are such that $0 \sim 1 \sim 2 \sim \cdots \sim 199 \sim 200$, while at the same time $200 \succ 0$. Obviously, these preferences violate transitivity.

Roadmap

Representing preferences

- The preference relation can also be **represented by numbers**, if we agree on the convention that *if the number assigned to an alternative* x *is not smaller (greater) than the number assigned to another alternative* y then x is at least as good as (better than) y.
  For instance, $V \succ M \sim L$ can be represented by assigning $V \rightarrow 3$, $M \rightarrow 1$, $L \rightarrow 1$.
- Such an assignment is called a **utility function.**

Representing preferences

Definition

We say that a function $u : X \to \mathbb{R}$ is a **utility representation** of the preferences $\succsim$ if for any pair $(x, y) \in X \times X$ it is the case that

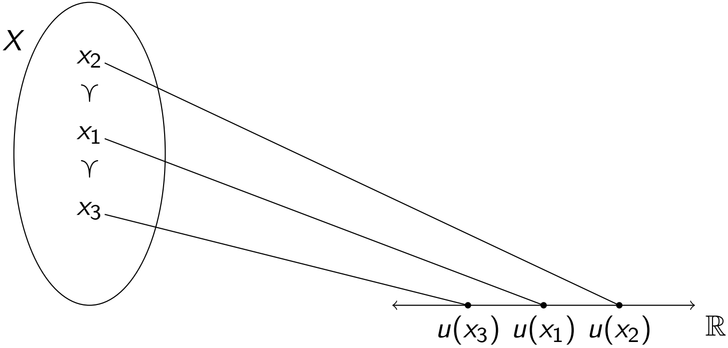
$$x \succsim y \;\Leftrightarrow\; u(x) \geq u(y).$$

Then, we say that u is a utility function.

- The primitive concept here is the preferences and not the utility function.
- A utility function is an abstract object used only to represent preferences without having a specific meaning.
- Note that $u(V) = 3$ and $u(L) = 1$ should <u>not</u> be interpreted as, the utility from vacation is three times as large as the utility from a lottery ticket. Utility has only *relative* meaning.

Intuition

- Binary relations are in general complex objects.
- It is much easier to work with real numbers.
- We associate each alternative with a real number (via the function u).
- Instead of comparing alternatives, we compare the corresponding utilities.



A simple example of a utility representation

- Take the weak preference relation from the previous example.
- Define the function $u : X \to \mathbb{R}$ by

$$u(M) = 1 \quad u(L) = 1 \quad u(V) = 3$$

- This is a utility representation:
  - $V \succsim M \Leftrightarrow u(V) \geq u(M)$
  - $L \not\succsim V \Leftrightarrow V \succ L \Leftrightarrow u(V) > u(L)$
  - $M \succsim L \Leftrightarrow u(M) \geq u(L)$

$\succsim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	✓	✓	✓

Existence

Does a utility function always exist?

Proposition

If preferences are a weak order over a finite set of alternatives, then there exists a utility representation.

The proof relies on the following intermediate result.

Lemma

If preferences are a weak order over a finite set of alternatives, then there is a least-preferred and a most-preferred alternative in this set.

Uniqueness

If a utility function exists, is it unique?

- Recall the utility representation $u : X \to \mathbb{R}$, defined by
  $$u(M) = 1 \quad u(L) = 1 \quad u(V) = 3$$
- Take the function $v : X \to \mathbb{R}$, defined by
  $$v(M) = 4 \quad v(L) = 4 \quad v(V) = 8$$
- This is another utility representation:
  - $V \succsim M \Leftrightarrow v(V) \geq v(M)$
  - $L \not\succsim V \Leftrightarrow v(L) < v(V)$
  - $M \succsim L \Leftrightarrow v(M) \geq v(L)$

$\succsim$	Money	Lottery	Vacation
Money	✓	✓	X
Lottery	✓	✓	X
Vacation	✓	✓	✓

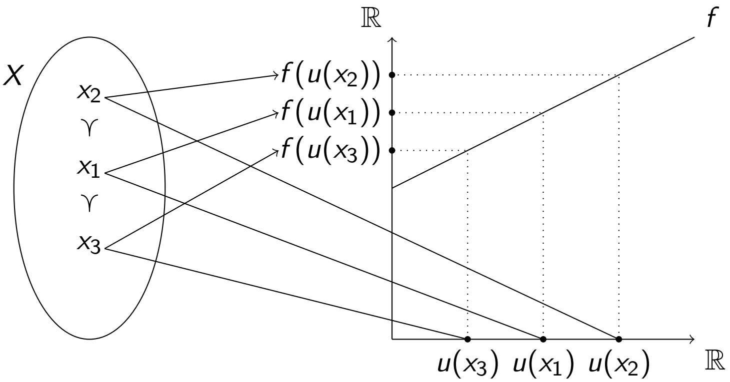
Monotonic transformation

Proposition

Let X be a finite set of alternatives. Suppose that $u : X \to \mathbb{R}$ is a utility representation of $\succsim$. Then, for every strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, the function $(f \circ u) : X \to \mathbb{R}$ is also a utility representation of $\succsim$.

Graphical representation of the transformation result

- Let u be a utility representation.
- Take a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$.
- Find $f(u(x))$ for each $x \in X$. This is also a utility representation.



Roadmap

From preferences to decisions

The second building block is **choices/actions**.

- **Choice set:** The set of choices (else called actions) is a subset $C \subseteq X$ of the set of alternatives.
  - The set of alternatives contains all situations that could possibly happen, whereas the set of choices contains only the alternatives that the agent can actually pick.
  - An agent has preferences over the entire set of alternatives. Thus, the domain of the utility function is X and not just C.
  - The choice set may be thought of as what the decision maker can afford, while the set of alternatives is all s/he desires. "You can't always get what you want." (from *Let it bleed* of the Rolling Stones (1968))

Decision problem & rational choice

Definition

A **decision problem** consists of a preferences relation over a set of alternatives $(X, \succsim)$ and a choice set $C \subseteq X$.

Definition

Consider a decision problem $\big((X, \succsim), C\big)$. We say that $x \in C$ is a **rational/optimal choice** if it is the case that $x \succsim y$ for all $y \in C$.

Existence of rational choice

Proposition

Consider a decision problem $((X, \succsim), A)$, such that A is a compact set. Moreover, assume that there exists a continuous utility representation $u : X \to \mathbb{R}$. Then, there exists an optimal choice $a \in A$.

Proof.

It follows directly from the extreme value theorem. According to it, a continuous function on a compact set achieves a maximum. □

Roadmap

Intertemporal choice problem

- For many decisions, costs are incurred and benefits are obtained in different points in time.
- Requires comparing one's welfare at some point in time t with one's welfare at some later point in time $t' > t$.
- This defines an **intertemporal choice problem**
- Humans often favor present rewards over later ones. `https://www.youtube.com/watch?v=QX_oy9614HQ`

Present value and discounting

What would you choose when I offer you

- €100 today vs. €100 in a year from today
- €100 today vs. €110 in a year from today
- €100 today vs. €500 in a year from today

A slight modification of the problem. **How would you choose between**

- €B a year from today, and
- €A today with the constraint that you cannot spend it until a year has passed?

Present value and discounting

- Suppose you take €A today. You may want to deposit it on an interest-bearing account with, say, interest rate r.
- After a year the initial €A have increased (hopefully) by €Ar
- Thus, in a year your balance will be €$(A + Ar) =$ €$A(1 + r)$, which is the **future value of** A **today**.

Answer to above question:

- Choose A today if $A(1 + r) > B$ and choose B a year from today if $A(1 + r) < B$
- Put differently, $(A, today) \succ (B, year\ from\ today)$ if and only if $A(1 + r) > B$

Present value and discounting

- Have seen $A(1+r)$ is the future value of A today.
- Alternatively, but equivalent, consider the **present value** of B: this is the money amount x received today and put on an account with interest rate r that would yield exactly B in a year from today:

$$x(1+r) = B, \;\; \textit{that is } x = B\frac{1}{1+r}.$$

- r is **interest/discount rate**, $\delta = \frac{1}{1+r}$ is **discount factor**.
- Thus, choose A today if $A > \delta B$ and choose B a year from today if $A < \delta B$

Present value and discounting

- Now assume you receive from today onwards for 3 more years, every year €100 you are not allowed to consume but put on an account with interest rate r. What will be the present value of this stream of money?
- The 100 today: 100; the 100 in one year: $\delta 100$; the 100 in two years: $\delta^2 100$; the 100 in three years: $\delta^3 100$;
- The **present value of the whole money stream**:

$$100 + \delta 100 + \delta^2 100 + \delta^3 100.$$

Present value and discounting

Summary: Call today date 0, date t is t periods into the future, r is the interest rate, and $\delta = 1/(1+r)$ the discount factor then

- the present value of €B available at date t is

$$\delta^t B$$

- the present value of the sequence $(€B_0, €B_1, ..., €B_T)$, for every date $t = 0, 1, ..., T$ where B_t is the amount of money available at date t, is

$$\delta^0 B_0 + \delta^1 B_1 + \cdots + \delta^T B_T.$$

Dated outcomes and time preferences

Someone who prefers a 1-week vacation now, over a 2-week vacation in 3 months intuitively appears more impatient, but ...

... how can we model impatience?

- Recall: Without time, $x \succsim y$ means x is at least as good as y
- When introducing time, preferences are over pairs of an *outcome*, e.g., #-week vacation, and the *date* the outcome happens, e.g., now or in 3 months; that is, the *outcome is dated*.
- The dated outcome (z, t) means that outcome z is to be experienced/consumed at date t

Dated outcomes and time preferences

- We say, a decision maker at date 0 (i.e., today) has complete and transitive preferences $\succsim_0$ over pairs of outcomes and dates.
- $(z, t) \succsim_0 (z', s)$ means that, today, the decsision maker likes z at date t at least as much as z' at time s.
- Even more general: $(z, \hat{t}) \succsim_t (z', \bar{t})$ with $\hat{t}, \bar{t} \geq t$.
- Example: A decision maker who likes, today ($t = 0$), a 5-day vacation starting in 4 weeks ($\hat{t} = 4$) strictly more than a 3-day vacation starting in 1 week ($\bar{t} = 1$), can be characterized by

$$(5 \; days, 4) \succ_0 (3 \; days, 1)$$

- Note, that in a week it is perfectly possible that:

$$(3 \; days, 1) \succ_1 (5 \; days, 4)$$

Time preferences and exponential discounting

- Recall: Without time, the preference $x \succsim y$, can be represented with a utility function, $u(x) \geq u(y)$.
- The preference relation $\succsim_t$ at time t restricted to dated outcomes (z, t) is called *instantaneous ranking at date t*.
- We can have a *instantaneous utility function* that represents this instantaneous ranking, in the sense that

$$u_t(z) \geq u_t(z') \text{ if and only if } (z, t) \succsim_t (z', t).$$

Definition

The *exponential utility (or discounted utility)* model assumes that at date 0 the ranking $\succ_0$ can be represented by a utility function given by

$$U_0(z, t) = \delta^t u_t(z),$$

where δ $(0 < \delta \leq 1)$ is called the *discount factor* and $\delta = \frac{1}{1+\rho}$ where $\rho \geq 0$ is called (internal) *discount rate*.

Exponential discounting

- It follows that

$$(z, t) \succsim_0 (z', s) \text{ if and only if } \delta^t u_t(z) \geq \delta^s u_s(z').$$

- Example: Assume utility is linear in money M and does not change over time, that is, $u_t(M) = u_s(M) = M$, and that time is counted in weeks.
- When you need to decide today between receiving €90 in 52 weeks and €100 in 53 weeks.
  When are you going for the second option?

$$\delta^{52}€90 < \delta^{53}€100 \Leftrightarrow \frac{9}{10} < \delta$$

- When you need to decide today between receiving €90 today and €100 in 1 week.
  When are you going for the second option?

$$\delta^{0}€90 < \delta^{1}€100 \Leftrightarrow \frac{9}{10} < \delta$$

Time inconsistency and (quasi-)hyperbolic discounting

- It is commonly observed that $(€90, 52) \prec_0 (€100, 53)$ but $(€90, 52) \succ_{52} (€100, 53)$; that is, *time inconsistent* behavior in the form of *diminishing impatience* (also called *present bias*).
- Exponential discounting cannot account for this because it implies $\delta < 0.9$ and $\delta > 0.9$, which is impossible.
- In the **(quasi-)hyperbolic discounting** model a lower discount factor (i.e., more impatience) is assumed between present and near future and a higher discount factor (i.e., more patience) between the near future and more distant future.

From exponential to quasi-hyperbolic discounting

- A different way of writing the exponential discounting model:

$$U_0(z,t) = \begin{cases} \delta^0 u_o(z) = u_0(z) & \text{if } t = 0 \\ \delta^t u_t(z) & \text{if } t > 0 \end{cases}$$

- More generally, if $s \geq t$:

$$U_t(z,s) = \begin{cases} u_t(z) & \text{if } s = t \\ \delta^{s-t} u_s(z) & \text{if } s > t \end{cases}$$

.

From exponential to quasi-hyperbolic discounting

- Changing to the quasi-hyperbolic discounting model:

$$U_0(z,t) = \begin{cases} \delta^0 u_o(z) = u_0(z) & \text{if } t = 0 \\ \beta\delta^t u_t(z) & \text{if } t > 0 \end{cases}$$

- More generally, if $s \geq t$:

$$U_t(z,s) = \begin{cases} u_t(z) & \text{if } s = t \\ \beta\delta^{s-t} u_s(z) & \text{if } s > t \end{cases}$$

.

- Example: Set $\delta = 0.95$ and $\beta = 0.8$:
  First, $(€90, 52) \prec_0 (€100, 53)$ if and only if
  $\beta\delta^{52}90 < \beta\delta^{53}100$, i.e., $0.9 < \delta = 0.95$ ✓
  Second, $(€90, 52) \succ_{52} (€100, 53)$ if and only if
  $90 > \beta\delta^{53-52}100$, i.e., $0.9 > 0.8 * 0.95 = 0.76$ ✓

Hyperbolic discounting

- Hyperbolic discounting model:

$$U_0(z,t) = \begin{cases} u_0(z) & \text{if } t = 0 \\ \frac{1}{1+kt} u_t(z) & \text{if } t > 0 \end{cases}$$

- More generally, if $s \geq t$:

$$U_t(z,s) = \begin{cases} u_t(z) & \text{if } s = t \\ \frac{1}{1+k(s-t)} u_t(z) & \text{if } s > t \end{cases}$$

.

- Suppose $(€50, 0) \succ_0 (€100, 6)$ and $(€50, 3) \prec_0 (€100, 9)$.
  Exponential discounting cannot account for this.
  Hyperbolic discounting can: Let $k = 0.25$ then
  $50 > 100/(1 + 0.25 * 6) = 40$✓ and
  $28.6 = 50/(1 + 0.25 * 3) < 100/(1 + 0.25 * 9) = 30.8$✓.

The end.