Expected utility

Economic Theory of Individual Decisions under Risk

Lecture 2
Neuroeconomics: An interdisciplinary approach to how the brain makes us deciding

Fall 2019

Roadmap

The elements of Decision Theory under Risk

The main building block is **lotteries** over **outcomes**.

- **Outcomes:** The set of outcomes is a finite set Z.
  - Examples: consumption bundles, monetary payoffs, etc.
- **Lotteries:** A lottery is a **probability distribution** over a **list of outcomes** Z.
  - Think, for instance, of a physical device assigning likelihoods to outcomes. Examples: tossing a coin, rolling a die, drawing from an urn, etc.
  - Importantly, the probability of each outcome being drawn is exogenously specified and assumed to be *objectively* known by the decision maker. Thus, we say that a lottery induces **objective uncertainty** or **risk**.

Representation of lotteries

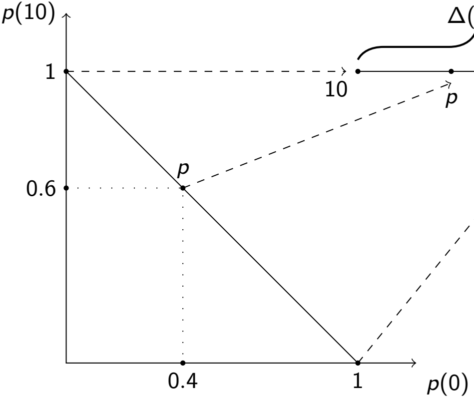
- Formally, a lottery is a probability distribution over Z, with $\Delta(Z)$ denoting the set of all lotteries.
- A lottery $p \in \Delta(Z)$ is identified by the probability of each different outcome being drawn:

$$\boxed{\left(p(z_1) \circledast z_1, \ldots, p(z_n) \circledast z_n\right)}$$

- Thus, the following two random experiments correspond to the same lottery, $(0.5 \circledast 10, 0.5 \circledast 0)$.
  - **Toss a coin**. In case of Heads you get 10 Euros; in case of Tails you get 0 Euros.
  - **Roll a die**. In case of 1-3 you get 10 Euros; in case of 4-6 you get 0 Euros.
- Every (certain) outcome $z \in Z$ is a (degenerate) lottery $(1 \circledast z)$.

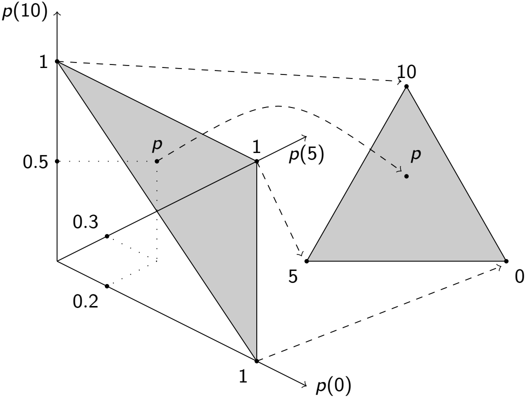
Lotteries over two outcomes - graphical representation

- The set of outcomes is $Z = \{0, 10\}$.
- Lotteries: All $\big(p(0), p(10)\big)$ such that $p(0) + p(10) = 1$, e.g., $p(0) = 0.4, p(10) = 0.6$.
- One linear segment suffices to represent $\Delta(Z)$.



Lotteries over three outcomes - graphical representation

- The set of outcomes is $Z = \{0, 5, 10\}$.
- Lotteries: All $\big(p(0), p(5), p(10)\big)$ such that $p(0) + p(5) + p(10) = 1$, e.g., $p(0) = 0.2, p(5) = 0.3, p(10) = 0.5$.
- One triangle suffices to represent $\Delta(Z)$.



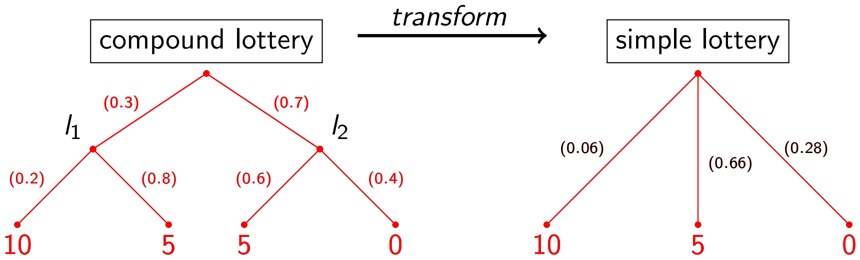
Compound lotteries

- A **compound lottery** is a lottery over lotteries.
- The compound lottery $(\alpha_1 \circledast p_1, \ldots, \alpha_m \circledast p_m) \in \Delta(\Delta(Z))$ is reduced to the simple lottery

$$\boxed{(\alpha_1 \circledast p_1, \ldots, \alpha_m \circledast p_m)(z) := \alpha_1 \cdot p_1(z) + \cdots + \alpha_m \cdot p_m(z)}$$

Compound lotteries and simple lotteries

- Let $l_1 = (0.2 \circledast 10, 0.8 \circledast 5)$ and $l_2 = (0.6 \circledast 5, 0.4 \circledast 0)$.
- Take $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$.



- $(0.3 \circledast l_1, 0.7 \circledast l_2)(10) = 0.3 \cdot 0.2 = 0.06$
- $(0.3 \circledast l_1, 0.7 \circledast l_2)(5) = 0.3 \cdot 0.8 + 0.7 \cdot 0.6 = 0.66$
- $(0.3 \circledast l_1, 0.7 \circledast l_2)(0) = 0.7 \cdot 0.4 = 0.28$

$$\boxed{(0.3 \circledast l_1, 0.7 \circledast l_2) = (0.06 \circledast 10, 0.66 \circledast 5, 0.28 \circledast 0)}$$

Roadmap

St. Petersburg Paradox (1)

- Consider the **following lottery**, where a fair coin is tossed (possibly) repeatedly:
  - First toss: if tail appears the game ends and you go home with €0, if head appears €2 are put in the pot and the coin is tossed again →
  - Second toss: if tail appears the game ends and you go home with the €2 in the pot, if head appears €4 are added in the pot and the coin is tossed again →
  - Third toss: if tail appears the game ends and you go home with the €6 in the pot, if head appears €8 are added in the pot and the coin is tossed again → ...
  - Fourth toss ... n-th toss ...: ... → ... → ...
- Question: What *sure payment* in € would you be *maximally willing to pay* to me to play this *lottery*?

St. Petersburg Paradox (2)

- What is the **expected value** (EV) of this lottery?

$$EV = \frac{1}{2}\times 2+\frac{1}{2}\times\frac{1}{2}\times 4+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times 8+\cdots = \frac{1}{2}\times 2+\frac{1}{4}\times 4+\frac{1}{8}\times 8+\cdots = 1+1+1+\cdots=\infty$$

- Unlikely that anyone would be willing to pay up to ∞ to play this lottery. Thus, EV appears **not to be a good way to represent a decision maker's preferences over lotteries** (first described by Daniel and Nicolas Bernoulli).
- Example of **expected utility** function: $u(x) = \ln x$.

$$EU(2) = \sum_{n=1}^{\infty}\frac{\ln 2^n}{2^n} = \sum_{n=1}^{\infty}\frac{n\ln 2}{2^n} = \ln 2\sum_{n=1}^{\infty}\frac{n}{2^n} = (\ln 2)\times 2 \approx 1.4 < \infty$$

- Can we say something more general? A **more general utility function**?

Lotteries as alternatives

- We take the set of lotteries to be the set of alternatives, i.e.,

$$\boxed{X := \Delta(Z)}$$

- Then, we consider a decision maker with preferences over $\Delta(Z)$ still denoted by $\succsim$.

von Neumann-Morgenstern axioms

Definition

We say that an agent has **von Neumann-Morgenstern (vNM) preferences** over $\Delta(Z)$ whenever $\succsim$ satisfies the following *axioms*:

(A_1') **Completeness**: For any two lotteries $p, q \in \Delta(Z)$, it is the case that $p \succsim q$ or $q \succsim p$.

(A_2') **Transitivity**: For any three lotteries $p, q, r \in \Delta(Z)$ with $p \succsim q$ and $q \succsim r$, it is the case that $p \succsim r$.

(A_3') **Continuity**: For any three lotteries $p, q, r \in \Delta(Z)$ with $p \succ q \succ r$, there exists some $\alpha \in (0, 1)$ such that $q \sim \big(\alpha \circledast p, (1-\alpha) \circledast r\big)$.

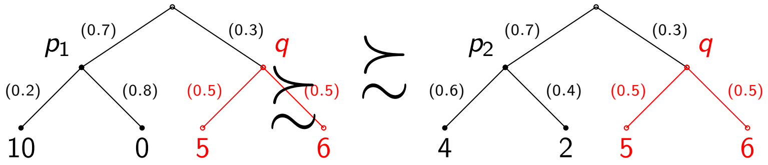
(A_4') **Independence of irrelevant alternatives (IIA)**: For any three lotteries $p, q, r \in \Delta(Z)$ and any $\alpha \in [0, 1]$, it is the case that $p \succsim q$ if and only if $\big(\alpha \circledast p, (1-\alpha) \circledast r\big) \succsim \big(\alpha \circledast q, (1-\alpha) \circledast r\big)$.

Continuity

- Take three monetary outcomes $Z = \{0, 5, 10\}$ and assume that $10 \succ 5 \succ 0$.
- Draw the set of lotteries over $\{0, 10\}$.
- Assume that the agent strictly prefers the 5 Euros over every red lottery, and strictly prefers every green lottery over the 5 Euros.
- Continuity says that the point where the green and the red segment meet is neither red nor green, but blue (indifference). The 'gap' between the red and the blue lotteries can be 'filled' by another lottery.

Independence of irrelevant alternatives

- Let $p_1 = (0.2 \circledast 10, 0.8 \circledast 0)$ and $p_2 = (0.6 \circledast 4, 0.4 \circledast 2)$.
- Mix both with another lottery $q = (0.5 \circledast 5, 0.5 \circledast 6)$.
- The preference relation has the same direction *with* the lottery q and *without* the lottery q.



Important properties of vNM preferences

Lemma

Consider a finite set of outcomes Z and suppose that the preferences $\succsim$ over $\Delta(Z)$ satisfy $A'_1 - A'_4$. Then, the following hold for all $p, q \in \Delta(Z)$ and $\alpha, \beta \in \mathbb{R}$:

(i) *If $p \succ q$ and $\alpha > \beta$, then* $\big(\alpha \circledast p, (1-\alpha) \circledast q\big) \succ \big(\beta \circledast p, (1-\beta) \circledast q\big)$.

(ii) *If $p \sim q$ and $\alpha \in [0,1]$, then* $\big(\alpha \circledast p, (1-\alpha) \circledast q\big) \sim q$.

Illustration Part (i) of the Lemma

$$p \succ q \text{ and } \alpha > \beta \Rightarrow \big(\alpha \circledast p, (1-\alpha) \circledast q\big) \succ \big(\beta \circledast p, (1-\beta) \circledast q\big)$$

- Let $p = (0.2 \circledast 10, 0.8 \circledast 0)$ and $q = (0.6 \circledast 4, 0.4 \circledast 2)$.
- Take $\alpha = 0.7$ and $\beta = 0.4$.

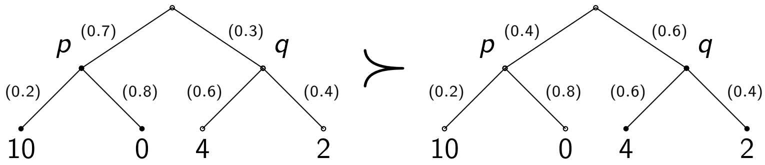
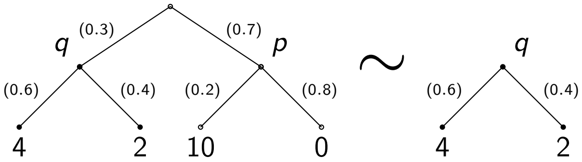


Illustration Part (ii) of the Lemma

$$p \sim q \text{ and } \alpha \in [0,1] \Rightarrow \big(\alpha \circledast p, (1-\alpha) \circledast q\big) \sim p$$

- Let $p = (0.2 \circledast 10, 0.8 \circledast 0)$ and $q = (0.6 \circledast 4, 0.4 \circledast 2)$.
- Take $\alpha = 0.7$.



Roadmap

Representing vNM preferences

Can we represent vNM preferences with a utility function?

Remember, proposition from previous lecture:

Proposition

If preferences are a weak order over a finite set of alternatives, then there exists a utility representation.

- Unfortunately, this does not imply that there also exists a utility representation for preferences over lotteries.
- The reason is that the *set of lotteries*, $\Delta(Z)$, is *not finite*. In fact, $\Delta(Z)$ has an enormous number of alternatives, which makes it difficult to find a representation.

Expected utility theorem

Theorem (von Neumann and Morgenstern, 1944)

Consider a finite set of outcomes Z. Then, the preferences $\succsim$ over $\Delta(Z)$ satisfy $A'_1 - A'_4$ if and only if there is a function $u : Z \to \mathbb{R}$ such that for every $p, q \in \Delta(Z)$,

$$p \succsim q \;\Leftrightarrow\; \sum_{z \in Z} p(z) \cdot u(z) \geq \sum_{z \in Z} q(z) \cdot u(z).$$

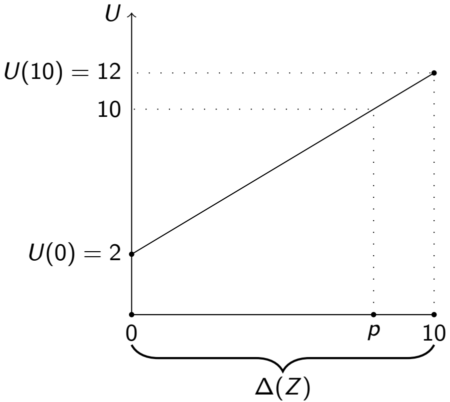
Moreover, $u : Z \to \mathbb{R}$ is unique up to a positive affine transformation.

The **(objective) expected utility** function that represent $\succsim$ is

$$\boxed{U(p) := \sum_{z \in Z} p(z) \cdot u(z)}$$

Example: Expected utility theorem with two outcomes

- Take $Z = \{0, 10\}$.
- Expected utility function:
  - Assign a utility $u(z)$ to each $z \in Z$.
  - Assign expected utility $U(p)$ to each $p \in \Delta(Z)$.



Positive affine transformations

- A positive affine transformation consists of (i) multiplying with a positive number, and (ii) adding any real number.
- In our context, a positive affine transformation of $u : Z \to \mathbb{R}$, is $v(z) := a + b \cdot u(z)$ for $b > 0$ and $a \in \mathbb{R}$.

Theorem

Let $U(p) = \sum_{z \in Z} p(z)u(z)$ represent the be vNM preferences $\succsim$ over $\Delta(Z)$. Then, $V(p) = \sum_{z \in Z} p(z)v(z)$ also represents $\succsim$.

Importance and limitations of Expected Utility Theorem

- The **Expected Utility Theorem** is one of the most influential results in the history of economics.
- This is because:
  - It established the axiomatic approach to Decision Theory.
  - The axioms are normatively appealing (are they?).
  - Game Theory (see next lecture) would not exist without it.
- Although, axioms are normatively appealing human decision makers (sometimes/often) violate them.

Violation of IIA: The Allais paradox

- Choose between:
  - $p_1 = (0.25 \circledast 3000, 0.75 \circledast 0)$
  - $p_2 = (0.2 \circledast 4000, 0.8 \circledast 0)$
- Choose between:
  - $q_1 = (1 \circledast 3000)$
  - $q_2 = (0.8 \circledast 4000, 0.2 \circledast 0)$
- In experiments most people choose $p_2 \succ p_1$ and $q_1 \succ q_2$.
- However observe:
  - $p_1 = (0.25 \circledast q_1, 0.75 \circledast 0)$
  - $p_2 = (0.25 \circledast q_2, 0.75 \circledast 0)$
- If IIA holds, then it must be $p_1 \succ p_2 \Leftrightarrow q_1 \succ q_2$. This is not what we observe in experiments.
- IIA is **often** violated.

Gains and losses

- Choose between:
  $p_1 = (1 \circledast 50),\ p_2 = (0.5 \circledast 100, 0.5 \circledast 0)$
- Choose between:
  $q_1 = (1 \circledast -50),\ q_2 = (0.5 \circledast -100, 0.5 \circledast -0)$
- In experiments most people choose $p_1 \succ p_2$ and $q_2 \succ q_1$; a phenomenon commonly called **loss aversion**.
- It is possible for a decision maker with vNM preferences to make these choices. Thus, *expected utility is not necessarily violated*.
- **But**, this depends on the wealth level and a vNM decision maker *cannot make such choices for every wealth level*.

The end.